AD-A054 255 VIRGINIA UNIV CHARLOTTESVILLE DEPT OF MECHANICAL AND--ETC F/G 20/11
TRANSIENT RESPONSE OF CONTINUOUS VISCOELASTIC STRUCTURAL MEMBER--ETC(U)
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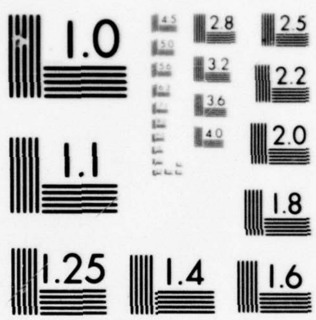
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TRANSIENT RESPONSE OF CONTINUOUS VISCOELASTIC STRUCTURAL MEMBERS

Technical Report

Contract No. N00014-75-C-0374

Submitted to:

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Arlington, Virginia 22217

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Report No. UVA/525303/MAE78/102

March 1978

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(14) REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: UVA/525303/MAE78/102

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER: (9)

4. TITLE (and Subtitle): (6) Transient Response of Continuous Viscoelastic Structural Members

5. TYPE OF REPORT & PERIOD COVERED: Technical Report.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): (10) W. D. Pilkey [redacted] J. Strenkowski

8. CONTRACT OR GRANT NUMBER(s): (15) N00014-75-C-0374

9. PERFORMING ORGANIZATION NAME AND ADDRESS: New
Department of Mechanical and Aerospace Engineering
University of Virginia
Charlottesville, Virginia 22901

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR 064-542/5-09-77 (474)

11. CONTROLLING OFFICE NAME AND ADDRESS:
Structural Mechanics Program
Office of Naval Research
Arlington, Virginia 22217

12. REPORT DATE: (11) March 1978

13. NUMBER OF PAGES

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): (12) 33p.

15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Transient Response
Modal Solutions
Viscoelastic Solids
Structural Members
Non-proportional Damping
Viscous Foundations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A comprehensive theory for the dynamic response of linear continuous viscoelastic structural members is formulated with a modal analysis. The constitutive relation is in the form of a hereditary integral. A general set of formulas is derived that may be used for both non-self-adjoint and self-adjoint systems of governing equations of motion. Applications include a Voigt-Kelvin beam and a viscoelastic circular plate.

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Introduction

In this paper a general formulation is presented for the dynamic response of a viscoelastic structural member using a modal approach. This formulation will depend only on the equations of motion of the structure. The formulation will apply to both self-adjoint and non-self-adjoint systems of equations of motion, non-homogeneous boundary and in-span conditions, and arbitrary excitation forces and displacements. In addition, proportional viscous damping may be included in a formulation.

The first to propose a modal solution of a viscoelasto-kinetic problem was Valanis [1]. By assuming that Poisson's ratio remains constant, the dynamic problem can be resolved into a quasi-static viscoelastic problem and a dynamic elastic problem. The latter problem can be solved using a classical modal analysis, while the quasi-static portion may be solved using a correspondence principle. Robertson and Thomas [2] used a similar philosophy, and arrived at slightly different results than Valanis. These were extensions of Valanis' results including more general boundary conditions and non-zero initial conditions. Robertson later applied these results to several specific members, such as a circular viscoelastic plate [3], and a viscoelastic beam [4]. However, the approach taken was not completely general, nor did it consider non-self-adjoint structural members.

In this paper, a general formulation is derived that goes beyond the work of the above investigators. A general set of formulas is derived that explicitly provides the relationship needed to determine the dynamic response of a viscoelastic member due to arbitrary loadings. A very general set of governing equations of motion can be uncoupled with this approach. These equations may be non-self-adjoint and possess general non-homogeneous boundary and in-span conditions. Two illustrations of a simple Voigt-Kelvin beam and an axisymmetric viscoelastic circular plate are given to demonstrate the use of the general formulation.

General Formulation

The dynamic response of a viscoelastic structural member will be formulated by using a modal analysis. The formulation will incorporate a damped viscous foundation, which must be proportional to the mass and/or stiffness of the member. The differential equations governing the linear motion of an isotropic and homogeneous viscoelastic member may be written as

$$(\lambda_0+\mu_0)G^* w_{k,ki} + \mu_0 G^* w_{i,kk} = \rho \frac{\partial^2 w_i}{\partial t^2} + c \frac{\partial w_i}{\partial t} \tag{1a}$$

with the boundary conditions,

$$\lambda_0 G^* w_{k,k} n_i + \mu_0 G^* (w_{i,k} + w_{k,i})\, n_k = p_i \qquad \text{on } B_1 \tag{1b}$$

$$w_i = 0 \qquad \text{on } B_2 \tag{1c}$$

and the initial conditions,

$$w_i(x,0) = w_{i0}(x); \qquad \frac{\partial w_i}{\partial t}(x,0) = \dot{w}_{i0}(x) \tag{1d}$$

where,

w_i = displacement in the i^{th} coordinate direction

λ_0, μ_0 = constants

n_j = unit outward normal to the surface

P_i' = prescribed surface tractions on B_1

$w_{i0}, \dot{w}_{i0}$ = prescribed initial displacement and velocity, respectively.

G = time-dependent relaxation modulus

In addition, the * notation denotes the convolution integral,

$$f(t)*g(t) = \int_{0-}^{t} f(t-\tau) \frac{\partial g(\tau)}{\partial \tau} d\tau \tag{2}$$

and $w_{i,k}$ denotes the partial derviative of w_i with respect to x_k. Note that in deriving these equations, it has been assumed that Poission's ratio remains constant. As Valanis points out, several researchers have shown that if a variation exists, it is not substantial, and that for incompressible materials (or for problems in which the dilation is not the primary mechanism of deformation) this simplification results in nearly exact solutions.

The equation of motion given by equations (1) may be written in the following alternate form:

$$G^* \, Du\,(x,t) = \sum_{j=1}^{2} A_j(x)\partial_j u(x,t) - F(x,t) \tag{3a}$$

with the initial conditions,

$$u(x,0) = u_0(x); \qquad \partial u(x,0) = \dot{u}_0(x) \tag{3b}$$

and the time-dependent boundary and in-span conditions,

$$G*L_i(u,t) = p_i(x,t) \quad \text{on } S \tag{3c}$$

where u(x,t) is a column vector of dependent state variables including displacements as well as internal forces. The notation in these equations is defined as follows:

$F(x,t)$ = N-dimensional column vector of body forces

$D(x), L_i(x)$ = NxN spatial matrix linear differential operators

$A_j(x)$ = NxN spatial matrix

$P_i(x,t)$ = N-dimensional column vector containing the prescribed non-homogeneous surface tractions and homogeneous prescribed displacements

$u_0(x), \dot{u}_0(x)$ = N-dimensional vectors of prescribed state variable initial conditions

$\partial_j = \partial^{(j)}/\partial t^{(j)}$

Note that external proportional viscous damping, such as found in a foundation, is included in the matrix $A_1(x)$. In this form, N equations of motion are represented by a matrix equation with differential operator elements. It must be kept in mind that these equations only apply to linear isotropic viscoelastic materials, deforming under isothermal conditions with a constant Poisson's ratio. Nevertheless, this general form of the equations of motion describes the response of a wide class of viscoelastic structural members.

The dynamic response of a general viscoelastic member may be expressed in the usual modal solution form,

$$u(x,t) = \sum_{m=1}^{\infty} h_m(t)\psi_m(x) \tag{4}$$

In this solution, $\psi_m(x)$ is a vector of undamped mode shapes corresponding to the vector of dependent state variables $u(x,t)$. The coefficient $h_m(t)$ is a scalar factor which indicates the relative contribution of the m^{th} mode to the overall response. In order to find the undamped mode shapes, the classical eigenvalue problem must be solved. Using the notation of equation (3), the undamped free motion problem takes the form

$$D\psi_n(x) = -(\lambda_n)^2 A_2(x)\psi_n(x) \tag{5a}$$

with the combined boundary and in-span conditions,

$$L_i\psi_n(x) = 0 \quad \text{on } S \tag{5b}$$

The viscoelastic member may be non-self-adjoint, so that a biorthogonality relation will be needed in deriving the elastic modal expansion. Hence, use will be made of the adjoint undamped free motion problem,

$$\tilde{D}\tilde{\psi}_n(x) = -(\lambda_n)^2 \tilde{A}_2(x)\tilde{\psi}_n(x) \tag{6a}$$

with the boundary and in-span conditions,

$$\tilde{L}_i\tilde{\psi}_n(x) = 0 \quad \text{on } S \tag{6b}$$

where the tildes refer to the algebraic adjoints of the corresponding quantities. The eigenfunctions, $\psi_n(x)$ and $\tilde{\psi}_n(x)$, can be shown to satisfy the biorthogonality relation [5] given by

$$\langle\tilde{\psi}_n, A_2\psi_m\rangle = \delta_{mn}Q_n \tag{7a}$$

and

$$Q_n = \langle\tilde{\psi}_n, A_2\psi_n\rangle \tag{7b}$$

where δ_{mn} is the kronecker delta and where the notation $\langle\tilde{\psi}_n, A_2\psi_n\rangle$ denotes the inner product,

$$\langle\tilde{\psi}_n, A_2\psi_n\rangle = \int_{\text{Domain}} \tilde{\psi}_n \cdot (A_2\psi_n)dx \tag{8}$$

Solution of equations (5) and (6) can be readily found for many structural members. Thus, only the temporal coefficients $h_m(t)$ must be determined to complete the modal solution in equation (4). To derive an uncoupled equation for these coefficients, begin with the extended Green's identity,

$$<\tilde{\psi}_m, D(G*u)> - <(G*u),\tilde{D}\tilde{\psi}_m> = B(\tilde{\psi}_m, (G*u)) \tag{9}$$

The term (G*u) is a function of both the spatial and temporal variables, which is assumed to be differentiable to the extent demanded by the operator D. The bilinear form $B(\tilde{\psi}_m,(G*u))$ represents the non-homogeneous boundary and in-span conditions. It is identified by forming the inner product $<\tilde{\psi}_m, D(G*u)>$ and then integrating by parts with respect to the spatial variables. Note that the appearance of G does not effect the integration by parts because G is a function of time. The boundary and in-span conditions are then grouped to form the bilinear functional $B(\tilde{\psi}_m,(G*u))$. Since the prescribed displacements on the boundary are zero, this bilinear form contains only the nonhomogeneous surface tractions which involve the relaxation modulus G(t). In fact, it is this form of $B(\tilde{\psi}_m,(G*u))$ that prohibits non-homogeneous displacements from being prescribed on the boundaries because these may not be expressed in the form G*u.

Before returning to equation (9), note that a property of the convolution is

$$G*(Du(x,t)) = (Du(x,t))*G \tag{10}$$

where Du(x,t) may be treated as an arbitrary function of x ant t. Since D is a spatial differential operator it does not effect G(a function of time) so that

$$G*Du - Du*G = D(u*G) = D(G*u) \tag{11}$$

With this equation the governing equation of motion (equation (3a)) becomes

$$D(G*u) = \sum_{j=1}^{2} A_j(x)\partial_j u(x,t) - F(x,t) \tag{12}$$

Now substitute equations (6a) and (12) into the Green's identity in equation (9) to obtain

$$\partial<\tilde{\psi}_m,A_1u> + \partial_2<\tilde{\psi}_m,A_2u> + \lambda_m^2 <(G*u), \tilde{A}_2\tilde{\psi}_m > \tag{13}$$

$$= <\tilde{\psi}_m,F> + B(\tilde{\psi}_m,(G*u))$$

Noting the homogeneous form of the Green's identity,

$$<(G*u), \tilde{A}_2\psi_m> = < \tilde{\psi}_m,A_2(G*u)> \tag{14}$$

and the G is a function of time, it follows that

$$<\tilde{\psi}_m,A_2(G*u)> = G*<\tilde{\psi}_m,A_2u> \tag{15}$$

or,

$$<(G*u),\tilde{A}_2\tilde{\psi}_m> = G*<\tilde{\psi}_m,A_2u> \tag{16}$$

Using equation (16) in equation (13) gives

$$\partial_2<\tilde{\psi}_m,A_2u> + \partial<\tilde{\psi}_m,A_1 \ddot{u}> + (\lambda_m)^2 G*<\tilde{\psi}_m,A_2u>$$

$$= <\tilde{\psi}_m,F> + B(\tilde{\psi}_m,(G*u) \tag{17}$$

Invoking the proportionality conditions,

$$<\tilde{\psi}_m, A_1 u> = (a + b\lambda_m{}^2) <\tilde{\psi}_m, A_2 u> - bB(\tilde{\psi}_m, u) \qquad (18)$$

a common inner product may be factored out of equation (17) to give

$$\ddot{\xi}_m(t) + (a + b\lambda_m^2)\dot{\xi}_m + (\lambda_m)^2 G{*}\xi_m$$

$$= <\tilde{\psi}_m, F> + B(\tilde{\psi}_m, (G{*}u)) + b\partial B(\tilde{\psi}_m, u) \qquad (19)$$

In this equation, a and b are constants of proportionality for viscous damping such that

$$A_1 u = aA_2 u - bDu \qquad (20)$$

and $\xi_m(t)$ is defined to be

$$\xi_m(t) = <\tilde{\psi}_m, A_2 u> \qquad (21)$$

so that the generalized initial conditions, $\xi_m(0)$ and $\dot{\xi}_m(0)$ are found from this equation at t = 0. The coefficients $h_m(t)$ are found by repeating the steps outlined in Reference [6] for elastic numbers. Since the orthogonality relation remains the same as for proportionally damped elastic members, it is not surprising that

$$h_m(t) = \frac{\xi_m(t)}{Q_m} \qquad (22)$$

which is identical to the result previously obtained.

Although a solution of the viscoelastic dynamic problem may in principle be given by equation (19), a more efficient form of the modal solution may be found by extracting a static solution from the modal expansion. This may be done be rearranging equation (19) into the form:

$$G*\xi_m = -\frac{1}{\lambda_m^2}\left[\ddot{\xi}_m + (a+b\lambda_m^2)\,\dot{\xi}_m - H_m\right] \tag{23}$$

where H_m is the generalized force term given by

$$H_m(t) = \int_x \tilde{\psi}_m F\,dx + B(\tilde{\psi}_m(G*u)) + b\partial B(\tilde{\psi}_m, u) \tag{24}$$

Taking the convolution of both sides of equation (4) and using equation (22) gives

$$G*u = \sum_{m=1}^{\infty} \frac{(G*\xi_m)}{Q_m}\,\psi_m(x) \tag{25}$$

Now introduce equation (23) into (25) and after some rearrangement obtain

$$G*u = \sum_{m=1}^{\infty} \frac{\psi_m H_m}{\lambda_m^2 Q_m} - \sum_{m=1}^{\infty} \frac{\psi_m}{\lambda_m^2 Q_m}\left[\xi_m + (a+b\lambda_m^2)\xi_m\right] \tag{26}$$

The first term in this equation is the static contribution to the viscoelastic solution. As Valanis [1] and Robertson and Thomas [2] have pointed out, a dynamic viscoelastic problem has a solution of the form

$$u(x,t) = \bar{v}_s(x,t) + \sum_{m=1}^{\infty} \frac{E_m(t)}{Q_m}\,\psi_m(x) \tag{27}$$

where $\bar{v}_s$ is the quasi-static viscoelastic solution. Equations (26) and (27) are equivalent if

$$G*\bar{v}_s = \sum_{m=1}^{\infty} \frac{\psi_m H_m}{\lambda_m^2 Q_m} \tag{28}$$

and

$$G*E_m = - \sum_{m=1}^{\infty} \frac{1}{\lambda_m^2} \left[\ddot{\xi}_m + (a+b\lambda_m^2)\dot{\xi}_m \right] \tag{29}$$

Then equation (26) becomes

$$G*u = G*\bar{v}_s + \sum_{m=1}^{\infty} (G*E_m) \frac{\psi_m}{Q_m} \tag{30}$$

which implies equation (27). It is now possible to obtain a more thorough understanding of the terms in equation (30). The Green's identity may be expressed in the form

$$<\tilde{\psi}_n, D(G*\bar{v}_s)> = <(G*\bar{v}_s), \tilde{D}\tilde{\psi}_n > + B(\tilde{\psi}_n, (G*\bar{v}_s)) \tag{31}$$

Substitute equations (6a) and (28) into this equation to obtain

$$<\tilde{\psi}_n, D(G*\bar{v}_s)> = - \sum_{m=1}^{\infty} \frac{\lambda_n^2}{\lambda_m^2 Q_m} \left[H_m(t) <\psi_m, \tilde{A}_2 \tilde{\psi}_n> \right] + B(\tilde{\psi}_n, G*\bar{v}_s)) \tag{32}$$

where integration of the series is assumed to be equal to a series of the integrals. Using the biorthogonality relation (equation (7)) and in view of equations (11) and (24), equation (32) becomes

$$<\tilde{\psi}_n, G*D\bar{v}_s> = -<\tilde{\psi}_n, F> - B(\tilde{\psi}_n, (G*u)) - b\partial(\tilde{\psi}_n, u) - B(\tilde{\psi}_n, (G*\bar{v}_s)) \tag{33}$$

The billinear forms in the above equation contain contributions to the generalized forcing function from the non-homogeneous tractions on the surface. In particular, the form $B(\tilde{\psi}_n, G*u))$ contains the non-homogeneous surface tractions as given by equation (3c). Therefore, $B(\tilde{\psi}_n, (G*\bar{v}_s))$ implies that the same boundary and in-span conditions apply to $\bar{v}_s$ as u when damping is absent (b=0). In addition, a differential equation valid throughout the volume is implied by the remaining two term in equation (33). That is, the following boundary value problem may be deduced from equation (33):

$$G{*}D(\bar{v}_s(x,t)) = -F(x,t) \tag{34a}$$

with boundary and in-span conditions

$$G{*}L_i\bar{v}_s(x,t) = P_i(x,t) \qquad \text{on } S \tag{34b}$$

The solution of this boundary value problem ($\bar{v}_s(x,t)$) will provide the quasi-static response of a viscoelastic member indicated in equation (27). Note that this solution is quasi-static in the sense that $\bar{v}_s$ is a function of x alone which is found for each instant of time.

To complete the dynamic response solution in equation (27), an uncoupled equation for $E_n(t)$ must be found. Substitute this equation into the governing equation of motion (equation (3a) to give

$$G{*}D\bar{v}_s + G{*}D\left(\sum_{m=1}^{\infty} E_m\psi_m\right) = A_1\dot{\bar{v}}_s + A_1\left(\sum_{m=1}^{\infty}\dot{E}_m\psi_m\right)$$
$$+ A_2\ddot{\bar{v}}_s + A_2\left(\sum_{m=1}^{\infty}\ddot{E}_m\psi_m\right) - F \tag{35}$$

where it is assumed that the infinite series converges at all points of interest. The first term in equation (35) is equal to -F which can then be cancelled from either side of this equation. Premultiply by $\tilde{\psi}_n$, then integrate over the volume to obtain

$$\langle\tilde{\psi}_n, A_1\dot{\bar{v}}_s\rangle + \sum_{m=1}^{\infty}\dot{E}_m\langle\tilde{\psi}_n, A_1\psi_m\rangle + \langle\tilde{\psi}_n, A_2\ddot{\bar{v}}_s\rangle$$
$$+ \sum_{m=1}^{\infty}\ddot{E}_m\langle\tilde{\psi}_n, A_2\psi_n\rangle = \sum_{m=1}^{\infty}(G{*}E_m)\langle\tilde{\psi}_n, D\psi_m\rangle \tag{36}$$

Note that the validity of reversing the convolution and the summation sign has been assumed. Apply equation (5a) to the above equation and then use the biorthogonality relation for undamped members (equation (7)) to find

$$\ddot{E}_m(t) + (a+b\lambda_m^2)\dot{E}_m + \lambda_m^2(G*E_m) = R_m(t) \tag{37a}$$

where

$$R_m(t) = -\frac{1}{Q_m}\left[\langle\tilde{\psi}_m, A_1\dot{\bar{v}}_s\rangle + \langle\tilde{\psi}_m, A_2\ddot{\bar{v}}_s\rangle\right] \tag{37b}$$

and where the proportionality condition in equation (18) has been employed. Note that the condition, $B(\tilde{\psi}_n, \psi_m) = 0$, which implies that all boundary and in-span conditions are homogeneous in the free motion problem, has also been used. The initial conditions for $E_m(t)$ may be found by premultiplying the initial condition form of equation (27) by $\tilde{\psi}_n A_2$ and then integrating over x to give

$$\langle\tilde{\psi}_n, A_2 u(x,0)\rangle = \langle\tilde{\psi}_n, A_2\bar{v}_s(x,0)\rangle + \sum_{m=1}^{\infty} \frac{E_m(0)}{Q_m}\langle\tilde{\psi}_n, A_2\psi_m\rangle \tag{38}$$

After applying biorthogonality, the generalized initial conditions are

$$E_m(0) = \left[\langle\tilde{\psi}_m, A_2 u_0(x)\rangle - \langle\tilde{\psi}_m, A_2\bar{v}_s(x,0)\rangle\right] \tag{39}$$

and similarly,

$$\dot{E}_m(0) = \langle\tilde{\psi}_m, A_2\dot{u}_0(x)\rangle - \langle\tilde{\psi}_m, A_2\dot{\bar{v}}_s(x,0)\rangle \tag{40}$$

An alternative form of the right-hand side of equation (37a) may be found which incorporates the complex compliance J(t) of the viscoelastic material. With this form, either the experimental or modeling information of a given material may be easily included in the solution for $E_m(t)$. Begin by applying the proportionality condition in equation (18) to give

$$<\psi_m, A_1\dot{\bar{v}}_s> = (a+b\lambda_m^2) <\tilde{\psi}_m, A_2\dot{\bar{v}}_s> - b\ B(\tilde{\psi}_m, \dot{\bar{v}}_s) \tag{41}$$

Substitution of this condition into equation (37b) leads to

$$R_m = -\frac{1}{Q_m}\left[(c_m\partial+\partial_2) <\tilde{\bar{\psi}}_m, A_2\bar{v}_s> - b\ B(\tilde{\psi}_m, \dot{\bar{v}}_s)\right] \tag{42}$$

where R_m denotes the right-hand side of equation (37a), and c_m is the proportionality constant $(a+\lambda_m^2 b)$. The inner product may now be transformed by use of the Green's identity into

$$<\tilde{\psi}_m, A_2\bar{v}_s> = <\bar{v}_s, \tilde{A}_2\tilde{\psi}_m> \tag{43}$$

From equation (6a), $\tilde{A}_2\tilde{\psi}_m$ may be found to be

$$\tilde{A}_2\tilde{\psi}_m = -\frac{\tilde{D}\tilde{\psi}_m}{\lambda_m^2} \tag{44}$$

which when substituted into equation (43) gives

$$<\tilde{\psi}_m, A_2\bar{v}_s> = -\frac{1}{\lambda_m^2} <\bar{v}_s, \tilde{D}\tilde{\psi}_m> \tag{45}$$

Now apply the extended Green's identity,

$$<\bar{v}_s, \tilde{D}\tilde{\psi}_m> = <\tilde{\psi}_m, D\bar{v}_s> + B(\tilde{\psi}_m, \bar{v}_s) \tag{46}$$

to obtain from equation (45) the result,

$$<\tilde{\psi}_m, A_2\bar{v}_s> = -\frac{1}{\lambda_m^2} <\tilde{\psi}_m, D\bar{v}_s> - \frac{1}{\lambda_m^2}\ B(\tilde{\psi}_m, v_s) \tag{47}$$

The term $D\bar{v}_s$ may now be transformed by taking the Laplace transform of equation (34a),

$$s\bar{G}D\bar{\bar{v}}_s = -\bar{F} \tag{48}$$

where the bar represents the corresponding transformed variable and

$$G{*}f = s\bar{G}\bar{f} \tag{49}$$

Equation (48) may be easily solved for $D\bar{\bar{v}}_s$ to obtain

$$D\bar{\bar{v}}_s = -\frac{\bar{\bar{F}}}{s\bar{G}} - s\bar{J}\bar{F} \tag{50}$$

where the relation between the Laplace transforms of the complex compliance $\bar{J}$ and the relaxation modulus $\bar{G}$ has been used, i.e.

$$\frac{1}{s\bar{G}} = s\bar{J} \tag{51}$$

Taking the inverse Laplace transform of equation (50) gives

$$D\bar{v}_s = -J * F \tag{52}$$

Finally, substitute this equation back into equation (47), which when placed in equation (42) leads to

$$R_m = -\frac{1}{Q_m}\left\{\frac{(c_m\partial+\partial_2)}{\lambda_m^2}\left[\langle\tilde{\psi}_m, J{*}F\rangle - B(\tilde{\psi}_m,\bar{v}_s)\right] - b\,B(\tilde{\psi}_m,\dot{\bar{v}}_s)\right\} \tag{53}$$

This is the alternative form for the right-hand side of equation (37a) which was sought. This form of $R_m(t)$ involves directly the complex compliance J of the viscoelastic material. When substituted back into equation (37a) the solution for $E_m(t)$ may now be found by taking several approaches. If a known viscoelastic model composed of springs and dashpots has been selected, the compliance will be known once the model parameters have been defined. These parameters may be prescribed a priori or chosen by fitting experimental data to the model [3]. In addition, experimental data may be used directly in equation (53) without any reliance on a particular model [4], in which case equation (37a) would be solved numerically. When a model is used, equation (37a) may be solved by Laplace transforms, which may be inverted analytically for simple models. An example of how this procedure may be carried out for a Voigt-Kelvin material will be shown in the following section. Otherwise, a numerical inversion process may be needed. Another approach which may be taken in finding the coordinates $E_m(t)$ from equation (37a) has been suggested by Valanis [1]. When a Laplace transform of equation (37a) has been taken, a Voltera integral equation of the second kind results. Then, any number of techniques may be used to solve this integral equation.

To summarize briefly, the dynamic response of a viscoelastic structural member on a proportional viscous foundation may be formally expressed by

$$u(x,t) = \bar{v}_s(x,t) + \sum_{m=1}^{\infty} \frac{E_m(t)}{Q_m} \psi_m(x) \tag{54}$$

where $\psi_m(x)$ and Q_m are the corresponding undamped elastic member eigenfunctions and norm, respectively. The temporal coefficients may be found by solving

$$\ddot{E}_m(t) + (a+b\lambda_m^2)\dot{E}_m + \lambda_m^2(G*E_m) = R_m(t) \tag{55}$$

and $R_m(t)$ may take on either of the two forms as given by equations (37b) or (53). As a special cases, when a = b = 0, a viscoelastic member without a damped foundation may be examined, and when the bilinear form is absent the member possesses homogeneous boundary and in-span conditions.

Illustrative Examples

The formulas derived in the previous section will be applied to a Voigt-Kelvin beam and an axisymmetric circular plate. The purpose in treating these examples is to demonstrate the details of the general formulation for a simple member and then to show that the response of a more complicated viscoelastic member may be uncoupled by applying the same steps used for simpler members. In addition, the theoretical results will be shown to compare with those obtained by Robertson [3], whose less general treatment was only valid for self-adjoint circular plates with no damping.

Dynamic Response of a Voigt-Kelvin Beam

Consider the undamped Euler-Bernoulli beam shown in Figure 1. It is assumed that the material may be modeled as a Voigt-Kelvin viscoelastic material. The linearly varying distributed load q(x,t) is applied at t = 0 and then maintained at that value for all time. That is,

$$q(x,t) = \frac{x}{\ell} H(t) \tag{56}$$

where H(t) is the Heaviside unit function, which possesses the property

$$H(t) * J(t) = J(t) \tag{57}$$

The dynamic deflection response for this beam is sought for this loading condition.

The governing equations of motion for a general viscoelastic beam have been given by Robertson [4]. The equations may be reduced to one fourth order differential equation when shear deformation and rotary inertia are neglected. When written in the form of equations (3), the following values are assigned:

$$D = 2(1 + \nu)\, I \frac{\partial^4}{\partial x^4} \qquad A_1 = 0 \tag{58a}$$

$$A_2 = -\rho \qquad F = -q \tag{58b}$$

$$u = w(x,t) \qquad u_0 = \dot{u}_0 = 0 \tag{58c}$$

$$(L_1)_{x=0,\ell} = 1 \qquad (L_2)_{x=0,\ell} \frac{\partial^2}{\partial x^2} \tag{58d}$$

$$(P_1)_{x=0,\ell} = (P_2)_{x=0,\ell} = 0 \tag{58e}$$

The dynamic response of this viscoelastic beam may be found from equation (27) to be

$$w(x,t) = w_s(x,t) + \sum_{m=1}^{\infty} \frac{E_m(t)}{Q_m} \psi_m(x) \tag{59}$$

where $w_s(x,t)$ is the quasi-static deflection. For this simple member, the deflected mode shapes $\psi_m(x)$ and corresponding eigenvalues λ_m may be analytically determined by solving equations (5). That is,

$$\psi_m(x) = \sin \frac{m\pi x}{\ell} \qquad m = 0, \pm 1, \pm 2, \ldots \tag{60}$$

and

$$\lambda_m = \left(\frac{m\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho}} \tag{61}$$

so that the corresponding norm may be explicitly determined from equation (7b) to be

$$Q_m = \rho \ell / 2 \tag{62}$$

Note that since the elastic beam is self-adjoint, only the modes $\psi_m(x)$ are needed to determine the response. Therefore, only the temporal coefficient remains to be determined in the series portion of equation (59). To find $E_m(t)$, recall from equations (37a) and (53) that

$$\ddot{E}_m(t) + \lambda_m^2 G{*}E_m = \frac{1}{\lambda_m^2 Q_m} \frac{\partial^2}{\partial t^2} \left[\int_0^\ell \psi_m (J{*}q)\, dx \right] \tag{63}$$

where a = b = 0 since the beam is undamped. In addition, the initial values, $E_m(0)$ and $\dot{E}_m(0)$ given by equations (39) and (40) are zero because the initial conditions u_0 and u_0 are zero. Introduce equations (56), (60), (61), and (62) into the above expression to arrive at

$$\ddot{E}_m + \lambda_m^2 G{*}E_m = \left(\frac{-2(-1)^m \ell^4}{(m\pi)^5 EI} \right) \ddot{J}(t) \tag{64}$$

where equation (57) has been utilized. Eliminate the time variable in this expression by taking the Laplace transform to obtain

$$s\,\bar{E}_m + \lambda_m^2 s\bar{G}\bar{E}_M = - \left(\frac{2(-1)^m \ell^4}{(m\pi)^5 EI} \right) s^2 \bar{J} \tag{65}$$

where the initial conditions of the complex compliance vanish since they are proportional to $E_m(0)$ and $\dot{E}_m(0)$ [37]. Solve for the Laplace transform of $E_m(t)$ (i.e. $\bar{E}_m(s)$) to obtain

$$\bar{E}_m = \frac{-s^3 \bar{J}\bar{J} L_m}{(\lambda_m^2 + s^3 \bar{J})} \tag{66}$$

where for convenience

$$L_m = \frac{2(-1)^m \ell^4}{(m\pi)^5 EI} \tag{67}$$

The Laplace transform of the complex compliance for a Voigt-Kelvin material [7] is given by

$$\bar{J} = \frac{1}{q_0}\left(\frac{1}{s} - \frac{1}{(\kappa + s)}\right) \tag{68}$$

where

$$\kappa = q_0/q_1 \tag{69}$$

and q_0 and q_1 are the spring and dashpot constants in the Voigt-Kelvin model. Introduce equation (68) into equation (66), and after some rearrangement arrive at the expression

$$\bar{E}_m = -L_m\left\{\frac{s}{(s-r_1)(s-r_2)} + \frac{\lambda_m^2}{\kappa}\left[\frac{1}{(s-r_1)(s-r_2)}\right] - \frac{1}{\kappa^2}\left[\frac{1}{s+\kappa}\right]\right\} \tag{70}$$

where

$$r_1, r_2 = \frac{-\lambda_m^2 q_1}{2} \pm \frac{\lambda_m}{2} \sqrt{q_1^2\lambda_m^2 - 4q_0} \tag{71}$$

and where the method of partial fractions has been utilized. From a table of Laplace transforms, $E_m(t)$ may be found by inverting equation (70). This results in

$$E_m(t) = -L_m\left[\frac{1}{\lambda_m\sqrt{\lambda_m^2 q_1^2 - 4q_o}}\left(r_1 e^{r_1 t} - r_2 e^{r_2 t}\right)\right.$$

$$\left. - \frac{\lambda_m}{\kappa\sqrt{\lambda_m^2 q_1^2 - 4q_o}}\left(e^{r_2 t} - e^{r_1 t}\right) - \frac{1}{\kappa^2} e^{-\kappa t}\right] \tag{72}$$

From this equation $E_m(t)$ may be determined. Returning to equation (59), only the quasi-static part of the response remains to be found. One technique that may be used is the correspondence principle [7]. Recall from elementary strength of materials that the static deflection of an elastic simply supported beam is

$$w(x) = \frac{q}{180\ EI}\ [3x^4 - 5x^2\ell^2 + 2\ell^4] \tag{73}$$

Using the correspondence principle, replace q and E by $(x/\ell)\ \frac{1}{s}$ and $\bar{Q}(s)/\bar{P}(s)$, respectively, to obtain

$$\bar{w}_s(x,s) = \frac{1}{180\ I\ell q_1}\ (\frac{1}{s(\kappa + s)})\ [3x^5 - 5x^3\ell^2 + 2x\ell^4] \tag{74}$$

which is the Laplace transform of the quasi-static deflection. Note that for a Voigt-Kelvin solid [7],

$$\frac{\bar{Q}(s)}{\bar{P}(s)} = q_1(\kappa + s) \tag{75}$$

This equation may be easily inverted to give for the quasi-static deflection response,

$$w_s(x,t) = (\frac{3x^5 - 5x^3\ell^2 + 2x\ell^4}{180\ I\ell q_0})\ \ (1 - e^{-\kappa t}) \tag{76}$$

which decays as time grows since $\kappa>0$. Hence, the solution of a Voigt-Kelvin beam given by equation (59) has been determined. It is apparent that many other methods of solution of equation (63) could have been pursued. For example, had J(t) been experimentally determined, a numerical solution of equation (63) would have been attempted. Or, this experimental data could be fitted to a more complicated but more realistic model. However, for more complicated viscoelastic models a drawback to the previous method of solution is that an inverse Laplace transform of a complicated function has to be taken. In these cases, a numerical inverse Laplace transform would probably be warranted.

<u>Dynamic Response of a Viscoelastic Circular Plate</u>

The previous example was given to demonstrate the usage of the general formulation to find the response of a simple member. The considerable potential of the general results will now be exhibited by considering the dynamic axisymmetric response of a viscoelastic annular circular plate with non-homogeneous moments and shears applied on the boundaries.

The equations of motion for a viscoelastic plate have been derived by Robertson [3]. In the format of equation (3), the following values are assigned:

$$D = \begin{bmatrix} \frac{D_o}{R_o^3}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\right) - \frac{h\kappa^2}{2a} & -\frac{h\kappa^2}{2R_o}\frac{\partial}{\partial R} \\ \frac{h\kappa^2}{2R_o}\left(\frac{\partial}{\partial R} + \frac{1}{R}\right) & \frac{h\kappa^2}{2R_o}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right) \end{bmatrix} \tag{77a}$$

$$A_2 = \rho h R_o \begin{bmatrix} \alpha^2 & 0 \\ 0 & 1 \end{bmatrix} \qquad F = \begin{Bmatrix} 0 \\ q \end{Bmatrix} \tag{77b}$$

$$u(R,t) = \begin{Bmatrix} \phi(r,t) \\ w(R,t) \end{Bmatrix} \qquad u_0(R) = \begin{Bmatrix} \phi_0(R) \\ w_0(R) \end{Bmatrix} \qquad \dot{u}_0(R) = \begin{Bmatrix} \dot{\phi}_0(R) \\ \dot{w}_0(R) \end{Bmatrix} \tag{77c}$$

The notation used in these equations is defined as follows:

ϕ, w = slope and radial non-dimensional deflection, respectively

ρ, κ, h, R_o = mass density, shear area correction factor, thickness, and outer radius of plate, respectively

R = non-dimensionalized variable radius, r/R_o where r is the plate radius

q = loading intensity on the radial face

$D_0 = h^3/\ 12(1 - \nu_0)$

$\alpha^2 = h^2/\ 12R_o^{\ 2}$

The non-homogeneous boundary conditions are given by equation (3c). That is, on $R = R_i$ (the inner radius) L_1 takes the form

$$L_1 = \begin{bmatrix} (D_0 \frac{\partial}{\partial R} + \frac{\nu_0}{R}) & 0 \\ 0 & 1 \end{bmatrix} \tag{78a}$$

and on $R = R_0$ (the outer radius)

$$L_2 = \frac{\kappa^2 h}{2} \begin{bmatrix} 1 & \frac{\partial}{\partial R} \\ 1 & 0 \end{bmatrix} \tag{78b}$$

Nothing that the bending moment along a circumference (M) and the transverse shear force on a radial face (Q) are given by

$$M = G^* D_0 \left(\frac{\partial \phi}{\partial R} + \nu_0 \frac{\phi}{R} \right) \tag{79}$$

$$Q = G^* \frac{\kappa^2 h}{2} \left(\phi + \frac{\partial w}{\partial R}\right) \tag{80}$$

equations (78) are seen to represent an applied moment and shear on the inner and outer plate radii, respectively. These applied loads are prescribed to be

$$P_1 = \begin{Bmatrix} f_1(t) \\ 0 \end{Bmatrix} \qquad P_2 = \begin{Bmatrix} f_2(t) \\ 0 \end{Bmatrix} \tag{81}$$

using the notation of equation (3c).

As in the previous example, the general formulas of the previous section may be used to construct the modal solution. For this member, the response is given by equation (27) as

$$\begin{Bmatrix} \phi(R,t) \\ \\ w(R,t) \end{Bmatrix} = \begin{Bmatrix} \bar{\phi}_s(R,t) \\ \\ \bar{w}_s(R,t) \end{Bmatrix} + \sum_{m=1}^{\infty} \frac{E_m(t)}{Q_m} \begin{Bmatrix} \phi_m(R) \\ \\ w_m(R) \end{Bmatrix} \tag{82}$$

where $\bar{\phi}_s$ and $\bar{w}_s$ are the quasi-static slope and deflection found by solving equations (34a) and (34b) for each instant of time. The norm Q_m and the mode shapes ϕ_m and w_m are found by solving the undamped free vibration problem given by equation (5). The temporal coefficients $E_m(t)$ are found from equation (37a) where a = b = 0 since no viscous foundation is present. The term $R_m(t)$ in equation (37a) is given by equation (53). Recall that the non-homogeneous boundary conditions contribute to the dynamic response through the bilinear form $B(\tilde{\psi}_m, \bar{\nu}_s)$ takes the form [5],

$$B(\tilde{\psi}_m, \bar{\nu}_s) = \{- \bar{\phi}_s M_m + \phi_m M - \bar{w}_s Q_m + w_m Q\} \, R/a \Big|_{R_i}^{R_o} \tag{83}$$

Therefore, $R_m(t)$ in equation (53) becomes

$$R_m = - \frac{1}{\lambda_m^2 Q_m} \frac{\partial^2}{\partial t^2} \{ \int_{R_i}^{R_o} w_m(J*q) R^{\frac{1}{2}} \, dR - w_m(R_i) f_2(t) \frac{R_i}{a} + \phi_m(R_0) f_1(t) \frac{R_0}{a} \} \tag{84}$$

These theoretical results are consistent with those obtained by Robertson [3] who used a less general approach. In that reference, the formulas presented here are numerically solved and the interested reader if referred to that account.

Summary

In this paper the dynamic response of a viscoelastic member was theoretically formulated. It has been shown that the uncoupled equation which results in this modal approach is an integro-differential equation. Solution of this equation is difficult; however, numerical solutions are possible [3], [4]. Note that this paper advances the past efforts of other authors in that non-self-adjoint systems of equation and proportional viscous damping may be readily included in the solution. Hence, the dynamic response of a very general viscoelastic structural member may be determined.

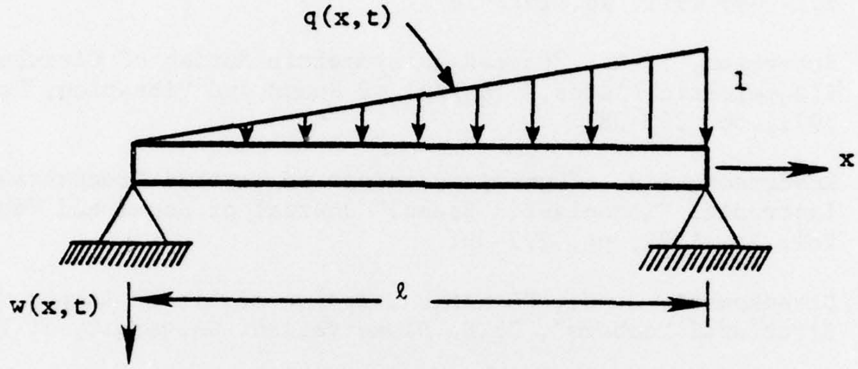

Figure 1 Voigt-Kelvin Beam Under Variable Loading

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NOTATION

a	constant of proportionality for viscous damping
A_j	N-square spatial matrix
b	constant of proportionality for viscous damping
B	bilinear functional
c_m	viscous proportionality constant, $c_m = a + \lambda_m^2 b$
D	N-square matrix linear differential operator
$\tilde{D}$	N-square algebraic adjoint matrix differential operator
D_0	$h^3/12(1 - \nu_0)$
E	modulus of elasticity
E_m	temporal coefficient
F	column vector of body forces
$G(t)$	relazation modulus for viscoelastic material
h	thickness of plate
h_m	temporal coefficient
$H(t)$	Heaviside unit function
H_m	generalized forcing function for structural member of viscoelastic material
I	moment of inertia
$J(t)$	complex compliance of a viscoelastic material
ℓ	length of beam
L_i	spatial matrix differential operator
M	bending moment
n_j	unit outward normal to bounding surface
P_i, P_i'	column vectors of non-homogeneous boundary and in-span conditions
q	applied loading intensity

Q	transverse shear force on a radial face
Q_m	classical norm
r	radial coordinate direction
R	non-dimensional radial coordinate direction
R_m	denotes right-hand side of viscoelastic equation
s	Laplace transform variable
S	denotes bounding surface and in-span condition locations
t	time
$u(x,t)$	column vector of dependent state variables
$\bar{v}_s$	column vector denoting the quasi-static response of a viscoelastic member
V	shear force
w	transverse beam deflection; radial circular plate deflection
w_i	displacement in the i^{th} coordinate direction
x	general point in the multidimensional region; one-dimensional coordinate direction in a rectangular coordinate system

Greek Symbols

α^2	$h^2/12a^2$
δ_{mn}	Kronecker delta
κ	q_0/q_1 in the Voigt-Kelvin model; shear area correction factor
λ_0	constant
λ_n	denotes the undamped (or classical) frequency or eigenvalue
μ	constant
ν_o	Poisson's ratio
ξ_m	temporal coefficient

π	3.14159...
ρ	mass density
σ_{rr}, σ_{zz}	normal stresses
τ	dummy time variable
τ_{rz}	shear stress
ϕ	slope of elastic curve for a circular plate
ψ_n	column vector of undamped (or classical) mode shapes or eigenfunctions corresponding to the nth frequency